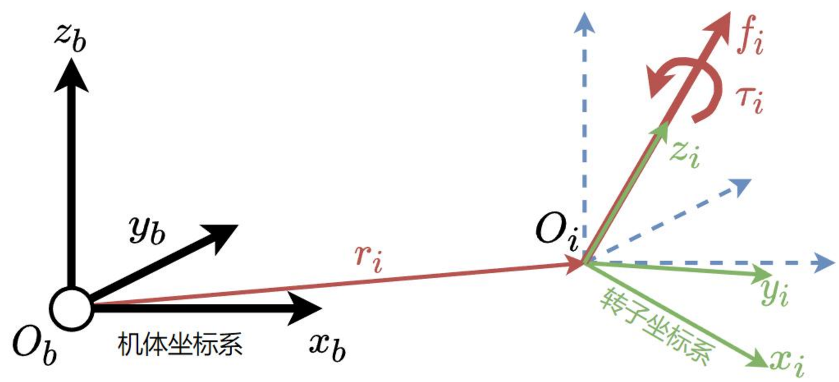
1.1 电机混控



无人机每个旋翼提供的升力和反作用力矩如图所示。定义第 i 个螺旋桨的转子坐标系为R_i，转子坐标系的 z 轴与电机轴重合。记第 i 个螺旋桨提供的升力和反作用力矩分别f_i和τ_i，对于逆时针(CCW)旋转的螺旋桨反作用力矩为负，对于顺时针(CW)旋转的桨反作用力矩为正。f_i和τ_i与螺旋桨转速ω_i的关系可以近似为：

$$\begin{aligned} f_i &= C_f\omega_i^2 \\ \tau_i &= aC_\tau\omega_i^2 \end{aligned} \tag{1-1}$$

其中C_f和C_τ是与空气动力学参数相关的系数。S_i表示螺旋桨的旋向，当螺旋桨逆时针(CCW)旋转时，$S_i =-1$;当螺旋桨 顺时针(CW)旋转时，$S_i = 1$。因此可以将τ_i记作：

$$\tau_i = S_i k f_i \tag{1-2}$$

其中 $k = C_\tau/C_f$。考虑到利用公式就可由升力求得对应的螺旋桨转速，为了简便书写，后续的推导将以螺旋桨的升力作为控制输入。

第 i 个螺旋桨在作用点处的力和力矩在机体坐标系下的表示为：

$$\begin{aligned} F_i &= R_i[0,0,f_i]^T \\ M_i &= R_i[0,0,S_i k f_i]^T \end{aligned} \tag{1-3}$$

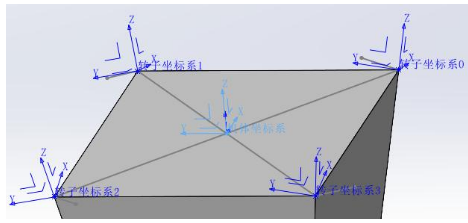
将所有力和力矩向质心简化，n 个螺旋桨提供的合力在机体坐标系下的表示为：

$$\begin{aligned} f &= \sum_{i=0}^{n-1} F_i \\ \tau &= \sum_{i=1}^{n-1}（r_i \times F_i + M_i） \end{aligned} \tag{1-4}$$

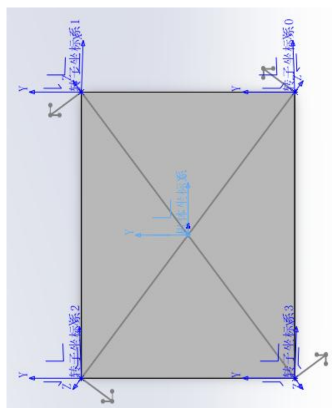
其中r_i为机体质心到螺旋桨受力点的位置矢量。解(1.11)和(1.12)，就可以由控制输入f和τ计算每个螺旋桨的升力和倾转电机的角度。

1.2 四棱锥模型

1.2.1 动力分配矩阵



控制可以简化为，底面为长方形的四棱锥。每个螺旋桨产生的力/力矩的作用点位于长方形的四个顶点，转子坐标系的 z 轴和四棱锥的棱共线。机体坐标系原点位于长方形几何中心，xoy 平面和长方形共面。



转轴单位向量公式（右手系，XY 平面内）：

$$u_0=\left(\frac{b}{\sqrt{a^2+b^2}},\frac{a}{\sqrt{a^2+b^2}},0\right) \tag{1-5}$$

$$u_1=\left(\frac{-b}{\sqrt{a^2+b^2}},\frac{a}{\sqrt{a^2+b^2}},0\right)$$

$$u_2=\left(\frac{-b}{\sqrt{a^2+b^2}},\frac{-a}{\sqrt{a^2+b^2}},0\right)$$

$$u_3=\left(\frac{b}{\sqrt{a^2+b^2}},\frac{-a}{\sqrt{a^2+b^2}},0\right)$$

各转子坐标系为标准坐标系绕各转轴旋转θ后的结果，转轴转角的 Rodrigues 公式为：

$$R(u,\theta)=\begin{pmatrix} u_x^2(1-cos\theta)+cos\theta & u_xu_y(1-cos\theta)-u_zsin\theta & u_xu_z(1-cos\theta)+u_ysin\theta \\ u_yu_x(1-cos\theta)+u_zsin\theta & u_y^2(1-cos\theta)+cos\theta & u_yu_z(1-cos\theta)-u_xsin\theta \\ u_zu_x(1-cos\theta)-u_ysin\theta & u_zu_y(1-cos\theta)+u_xsin\theta & u_z^2(1-cos\theta)+cos\theta \end{pmatrix} \tag{1-6}$$

其中，$u=\left(u_x,u_y,u_z\right)^T$，是单位向量。θ为转角。

首先，利用第一章中的公式来推导混控矩阵，可以列出以下关系：

$$F_0 = R_0[0,0,f_0]^T,\quad M_0 = R_0[0,0,S_0kf_0]^T,\quad r_0 = \left[\frac{a}{2},-\frac{b}{2},0\right]^T \tag{1-7}$$

$$F_1 = R_1[0,0,f_1]^T,\quad M_1 = R_1[0,0,S_1kf_1]^T,\ r_1 = \left[\frac{a}{2},\frac{b}{2},0\right]^T$$

$$F_2 = R_2[0,0,f_2]^T,\quad M_2 = R_2[0,0,S_2kf_2]^T,\quad r_2 = \left[-\frac{a}{2},\frac{b}{2},0\right]^T$$

$$F_3 = R_3[0,0,f_3]^T,\quad M_3 = R_3[0,0,S_3kf_4]^T,\quad r_3 = \left[-\frac{a}{2},-\frac{b}{2},0\right]^T$$

带入（1-4）可得：

$$f = \sum_{i=0}^{n-1} R_i[0,0,1]^T f_i \tag{1-8}$$

$$\tau = \sum_{i=1}^{n-1}（r_i \times R_i[0,0,1]^T + R_i[0,0,1]^T S_i k）f_i$$

$$\begin{bmatrix} f \\ \tau \end{bmatrix} = \begin{bmatrix} R_0[0,0,1]^T & R_1[0,0,1]^T & R_2[0,0,1]^T & R_3[0,0,1]^T \\ r_0 \times R_0[0,0,1]^T + R_0[0,0,1]^T S_0 k & r_1 \times R_1[0,0,1]^T + R_1[0,0,1]^T S_1 k & r_2 \times R_2[0,0,1]^T + R_2[0,0,1]^T S_2 k & r_3 \times R_3[0,0,1]^T + R_3[0,0,1]^T S_3 k \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{1-9}$$

所以：

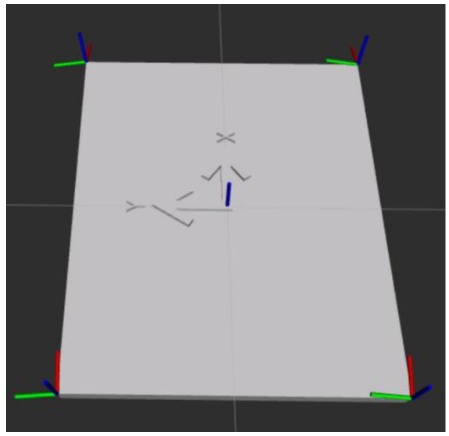
$$\begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}}sin\theta & \frac{a}{\sqrt{a^2+b^2}}sin\theta & -\frac{a}{\sqrt{a^2+b^2}}sin\theta & -\frac{a}{\sqrt{a^2+b^2}}sin\theta \\ -\frac{b}{\sqrt{a^2+b^2}}sin\theta & \frac{b}{\sqrt{a^2+b^2}}sin\theta & \frac{b}{\sqrt{a^2+b^2}}sin\theta & -\frac{b}{\sqrt{a^2+b^2}}sin\theta \\ cos\theta & cos\theta & cos\theta & cos\theta \\ -\frac{b}{2}cos\theta + \frac{S_0ka}{\sqrt{a^2+b^2}}sin\theta & \frac{b}{2}cos\theta + \frac{S_1ka}{\sqrt{a^2+b^2}}sin\theta & \frac{b}{2}cos\theta - \frac{S_2ka}{\sqrt{a^2+b^2}}sin\theta & -\frac{b}{2}cos\theta - \frac{S_3ka}{\sqrt{a^2+b^2}}sin\theta \\ -\frac{a}{2}cos\theta - \frac{S_0kb}{\sqrt{a^2+b^2}}sin\theta & -\frac{a}{2}cos\theta + \frac{S_1kb}{\sqrt{a^2+b^2}}sin\theta & \frac{a}{2}cos\theta + \frac{S_2kb}{\sqrt{a^2+b^2}}sin\theta & \frac{a}{2}cos\theta - \frac{S_3kb}{\sqrt{a^2+b^2}}sin\theta \\ S_0kcos\theta & S_1kcos\theta & S_2kcos\theta & S_3kcos\theta \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{1-10}$$

$$\begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = A \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{1-11}$$

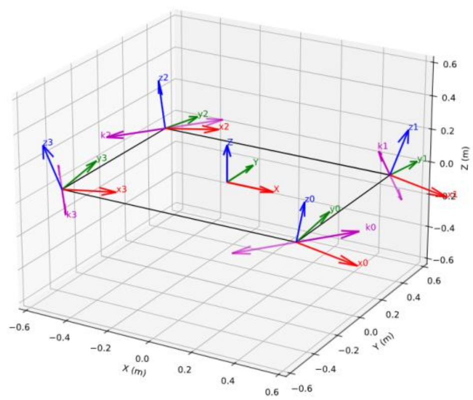
对上式求伪逆可得到：

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = A^+ \begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} \tag{1-12}$$

其中A^+表示A的伪逆矩阵。



Quadrotor corners frames and axes (theta=15.44°)



1.2.2 耦合证明

$$
\begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix}
\frac{a}{\sqrt{a^2+b^2}}\sin\theta & \frac{a}{\sqrt{a^2+b^2}}\sin\theta & -\frac{a}{\sqrt{a^2+b^2}}\sin\theta & -\frac{a}{\sqrt{a^2+b^2}}\sin\theta \\
-\frac{b}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{\sqrt{a^2+b^2}}\sin\theta & -\frac{b}{\sqrt{a^2+b^2}}\sin\theta \\
\cos\theta & \cos\theta & \cos\theta & \cos\theta \\
-\frac{b}{2}\cos\theta+\frac{S_0ka}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{2}\cos\theta+\frac{S_1ka}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{2}\cos\theta-\frac{S_2ka}{\sqrt{a^2+b^2}}\sin\theta & -\frac{b}{2}\cos\theta-\frac{S_3ka}{\sqrt{a^2+b^2}}\sin\theta \\
-\frac{a}{2}\cos\theta-\frac{S_0kb}{\sqrt{a^2+b^2}}\sin\theta & -\frac{a}{2}\cos\theta+\frac{S_1kb}{\sqrt{a^2+b^2}}\sin\theta & \frac{a}{2}\cos\theta+\frac{S_2kb}{\sqrt{a^2+b^2}}\sin\theta & \frac{a}{2}\cos\theta-\frac{S_3kb}{\sqrt{a^2+b^2}}\sin\theta \\
S_0k\cos\theta & S_1k\cos\theta & S_2k\cos\theta & S_3k\cos\theta
\end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{1-13}
$$

为了简化，假设旋转顺序为[+1,-1,+1,-1]。

$$
\begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix}
\frac{a}{\sqrt{a^2+b^2}}\sin\theta & \frac{a}{\sqrt{a^2+b^2}}\sin\theta & -\frac{a}{\sqrt{a^2+b^2}}\sin\theta & -\frac{a}{\sqrt{a^2+b^2}}\sin\theta \\
-\frac{b}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{\sqrt{a^2+b^2}}\sin\theta & -\frac{b}{\sqrt{a^2+b^2}}\sin\theta \\
\cos\theta & \cos\theta & \cos\theta & \cos\theta \\
-\frac{b}{2}\cos\theta+\frac{ka}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{2}\cos\theta-\frac{ka}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{2}\cos\theta-\frac{ka}{\sqrt{a^2+b^2}}\sin\theta & -\frac{b}{2}\cos\theta+\frac{ka}{\sqrt{a^2+b^2}}\sin\theta \\
-\frac{a}{2}\cos\theta-\frac{kb}{\sqrt{a^2+b^2}}\sin\theta & -\frac{a}{2}\cos\theta-\frac{kb}{\sqrt{a^2+b^2}}\sin\theta & \frac{a}{2}\cos\theta+\frac{kb}{\sqrt{a^2+b^2}}\sin\theta & \frac{a}{2}\cos\theta+\frac{kb}{\sqrt{a^2+b^2}}\sin\theta \\
S_0k\cos\theta & S_1k\cos\theta & S_2k\cos\theta & S_3k\cos\theta
\end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{1-14}
$$

$$\begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}}\sin\theta & \frac{a}{\sqrt{a^2+b^2}}\sin\theta & -\frac{a}{\sqrt{a^2+b^2}}\sin\theta & -\frac{a}{\sqrt{a^2+b^2}}\sin\theta \\ -\frac{b}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{\sqrt{a^2+b^2}}\sin\theta & -\frac{b}{\sqrt{a^2+b^2}}\sin\theta \\ \cos\theta & \cos\theta & \cos\theta & \cos\theta \\ -(\frac{b}{2}\cos\theta-\frac{ka}{\sqrt{a^2+b^2}}\sin\theta) & \frac{b}{2}\cos\theta-\frac{ka}{\sqrt{a^2+b^2}}\sin\theta & \frac{b}{2}\cos\theta-\frac{ka}{\sqrt{a^2+b^2}}\sin\theta & -(\frac{b}{2}\cos\theta-\frac{ka}{\sqrt{a^2+b^2}}\sin\theta) \\ -\frac{a}{2}\cos\theta-\frac{kb}{\sqrt{a^2+b^2}}\sin\theta & -\frac{a}{2}\cos\theta-\frac{kb}{\sqrt{a^2+b^2}}\sin\theta & -(-\frac{a}{2}\cos\theta-\frac{kb}{\sqrt{a^2+b^2}}\sin\theta) & -(-\frac{a}{2}\cos\theta-\frac{kb}{\sqrt{a^2+b^2}}\sin\theta) \\ S_0k\cos\theta & S_1k\cos\theta & S_2k\cos\theta & S_3k\cos\theta \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{1-15}$$

定义 4 个与力和力矩有关的组合量：

$$\begin{aligned} S &= f_0+f_1+f_2+f_3 \\ D_x &= f_0+f_1-f_2-f_3 \\ D_y &= -f_0+f_1+f_2-f_3 \\ \Sigma_s &= f_0-f_1+f_2-f_3 \end{aligned} \tag{1-16}$$

为把矩阵中的几行写成：

$$\begin{aligned} f_x &= \frac{a}{\sqrt{a^2+b^2}}sin\theta D_x \\ f_y &= \frac{b}{\sqrt{a^2+b^2}}sin\theta D_y \\ f_z &= cos\theta S \\ \tau_z &= kcos\theta\Sigma_s \end{aligned} \tag{1-17}$$

对于τ_x和τ_y：

$$\begin{aligned} \tau_x &= (\frac{b}{2}\cos\theta-\frac{ka}{\sqrt{a^2+b^2}}\sin\theta)D_y \\ \tau_y &= (-\frac{a}{2}\cos\theta-\frac{kb}{\sqrt{a^2+b^2}}\sin\theta)D_x \end{aligned} \tag{1-18}$$

所以：

$$\begin{aligned} \tau_x &= (\frac{b}{2}\cos\theta-\frac{ka}{\sqrt{a^2+b^2}}\sin\theta)\frac{\sqrt{a^2+b^2}}{bsin\theta}f_y \\ \tau_y &= (-\frac{a}{2}\cos\theta-\frac{kb}{\sqrt{a^2+b^2}}\sin\theta)\frac{\sqrt{a^2+b^2}}{asin\theta}f_x \end{aligned} \tag{1-19}$$

1.3 长方形模型

1.3.1 动力分配矩阵

令θ = 0：

$$\begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}}sin\theta & \frac{a}{\sqrt{a^2+b^2}}sin\theta & -\frac{a}{\sqrt{a^2+b^2}}sin\theta & -\frac{a}{\sqrt{a^2+b^2}}sin\theta \\ -\frac{b}{\sqrt{a^2+b^2}}sin\theta & \frac{b}{\sqrt{a^2+b^2}}sin\theta & \frac{b}{\sqrt{a^2+b^2}}sin\theta & -\frac{b}{\sqrt{a^2+b^2}}sin\theta \\ cos\theta & cos\theta & cos\theta & cos\theta \\ -\frac{b}{2}cos\theta + \frac{S_0ka}{\sqrt{a^2+b^2}}sin\theta & \frac{b}{2}cos\theta + \frac{S_1ka}{\sqrt{a^2+b^2}}sin\theta & \frac{b}{2}cos\theta - \frac{S_2ka}{\sqrt{a^2+b^2}}sin\theta & -\frac{b}{2}cos\theta - \frac{S_3ka}{\sqrt{a^2+b^2}}sin\theta \\ -\frac{a}{2}cos\theta - \frac{S_0kb}{\sqrt{a^2+b^2}}sin\theta & -\frac{a}{2}cos\theta + \frac{S_1kb}{\sqrt{a^2+b^2}}sin\theta & \frac{a}{2}cos\theta + \frac{S_2kb}{\sqrt{a^2+b^2}}sin\theta & \frac{a}{2}cos\theta - \frac{S_3kb}{\sqrt{a^2+b^2}}sin\theta \\ S_0kcos\theta & S_1kcos\theta & S_2kcos\theta & S_3kcos\theta \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{1-20}$$

$$\begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ -\frac{b}{2} & \frac{b}{2} & \frac{b}{2} & -\frac{b}{2} \\ -\frac{a}{2} & -\frac{a}{2} & \frac{a}{2} & \frac{a}{2} \\ S_0k & S_1k & S_2k & S_3k \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{1-21}$$

$$\begin{bmatrix} f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\frac{b}{2} & \frac{b}{2} & \frac{b}{2} & -\frac{b}{2} \\ -\frac{a}{2} & -\frac{a}{2} & \frac{a}{2} & \frac{a}{2} \\ S_0k & S_1k & S_2k & S_3k \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{1-22}$$

平移和姿态耦合。

1.4 加 2 垂直推进器 4，5 模型

1.4.1 动力分配矩阵

位于 4-[l,m,n],5-[l,-m,n]

$$f = \sum_{i=0}^{n-1} R_i[0,0,1]^T f_i \tag{1-23}$$

$$\tau = \sum_{i=1}^{n-1} (r_i \times R_i[0,0,1]^T + R_i[0,0,1]^T S_i k) f_i$$

$$\begin{gathered} R_{0-3} = I \\ R_{4-5} = Rot_y(90) \end{gathered} \tag{1-24}$$

$$r_0 = \left[\frac{a}{2}, -\frac{b}{2}, 0\right]^T r_1 = \left[\frac{a}{2}, \frac{b}{2}, 0\right]^T r_2 = \left[-\frac{a}{2}, \frac{b}{2}, 0\right]^T r_3 = \left[-\frac{a}{2}, -\frac{b}{2}, 0\right]^T r_4 = [l, m, n]^T r_5 = [l, -m, n]^T \tag{1-25}$$

$$\begin{bmatrix} f \\ \tau \end{bmatrix} = \begin{bmatrix} [0,0,1]^T & [0,0,1]^T & [0,0,1]^T & [0,0,1]^T & \mathrm{Rot}_y(90)[0,0,1]^T & \mathrm{Rot}_y(90)[0,0,1]^T \\ r_0 \times [0,0,1]^T + [0,0,1]^T S_0 k & r_1 \times [0,0,1]^T + [0,0,1]^T S_1 k & r_2 \times [0,0,1]^T + [0,0,1]^T S_2 k & r_3 \times [0,0,1]^T + [0,0,1]^T S_3 k & r_4 \times Rot_y(90)[0,0,1]^T + Rot_y(90)[0,0,1]^T S_4 k & r_5 \times Rot_y(90)[0,0,1]^T + Rot_y(90)[0,0,1]^T S_5 k \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} \tag{1-26}$$

$$
M=\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
-\frac{b}{2} & \frac{b}{2} & \frac{b}{2} & -\frac{b}{2} & S_4k & S_5k \\
-\frac{a}{2} & -\frac{a}{2} & \frac{a}{2} & \frac{a}{2} & n & n \\
S_0k & S_1k & S_2k & S_3k & -m & m
\end{bmatrix} \tag{1-27}
$$

$$
\begin{bmatrix} f_x \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix}=
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 \\
-\frac{b}{2} & \frac{b}{2} & \frac{b}{2} & -\frac{b}{2} & S_4k & S_5k \\
-\frac{a}{2} & -\frac{a}{2} & \frac{a}{2} & \frac{a}{2} & n & n \\
S_0k & S_1k & S_2k & S_3k & -m & m
\end{bmatrix}
\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} \tag{1-28}
$$

1.5 SO3 控制器

对于机器人位置：

$$
F_b - F_{\text{floating}} + G = F_{\text{d}} \tag{1-29}
$$

其中，F_b为世界系下作用到 base_link 的力，F_{floating}为浮力，G 为重力，F_d为世界系下推进器期望的合力。

设计 PD+前馈控制律为：

$$
\begin{aligned}
e_p &= x - x_{ref} \\
e_v &= v - v_{ref} \\
F_d &= -k_p e_p - k_d e_v + m a_{ref} - F_{floating} + m g e_z
\end{aligned} \tag{1-30}
$$

对于机器人姿态动力学：

$$
\begin{gathered}
\dot{R} = R\hat{w} \\
J\dot{w} + w \times (Jw) = \tau
\end{gathered} \tag{1-31}
$$

姿态误差:

$$
e_R = \frac{1}{2}(R_d^T R - R^T R_d)^{\vee} \tag{1-32}
$$

角速度误差:

$$
e_w = w - R_d^T R_d w_d \tag{1-33}
$$

控制律:

$$\tau = -k_R e_R - k_\Omega e_w + w \times (Jw) - J(\hat{w} R_d^\top R_d w_d - R_d^\top R_d \dot{w}_d) \tag{1-34}$$

所有都在机体坐标系下。